Minmax Methods for Geodesics and Minimal Surfaces

Tristan Rivière

ETH Zürich

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Lecture 2 : Palais Deformation Theory

in ∞ Dimensional Spaces.

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Example : Let l p > k

$$\mathcal{M} := W^{l,p}(\Sigma^k, N^n) := \left\{ \vec{u} \in W^{l,p}(\Sigma^k, \mathbb{R}^m) ; \ \vec{u}(x) \in N^n \text{ a.e. } x \in \Sigma^k \right\}$$

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then $\phi^{-1}(\overline{B_r(x)})$ might not be closed in \mathcal{M} .

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Proposition Let $(\mathcal{O}_{\alpha})_{\alpha \in A}$ be an arbitrary covering of a C^1 paracompact Banach manifold \mathcal{M} .

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where the sum is locally finite.

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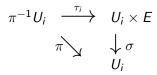
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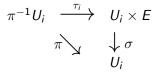
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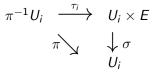


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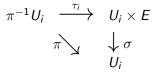
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$$x \in U_i \cap U_j \longrightarrow \tau_i \circ \tau_j^{-1}\Big|_{\pi^{-1}(x)} \in \mathcal{L}(E, E)$$

is C^p.

Finsler Structures on Banach Bundles.

Definition Let \mathcal{M} be a normal Banach manifold and let \mathcal{V} be a Banach Space Bundle over \mathcal{M} .

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Definition Let \mathcal{M} be a normal Banach manifold and let \mathcal{V} be a Banach Space Bundle over \mathcal{M} . A **Finsler structure** on \mathcal{V} is a continuous function

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Definition Let \mathcal{M} be a **normal** C^p Banach manifold. $T\mathcal{M}$ equipped with a Finsler structure is called a **Finsler Manifold**.

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Let Σ^2 be a closed oriented 2-dim manifold and N^n be a closed sub-manifold of \mathbb{R}^m .

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Let Σ^2 be a closed oriented 2-dim manifold and N^n be a closed sub-manifold of \mathbb{R}^m . Let q > 2

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The tangent space to ${\mathcal M}$ at a point $\vec{\Phi}$ is

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We equip $T_{\vec{\Phi}}\mathcal{M}$ with the following norm

$$\|\vec{v}\|_{\vec{\Phi}} := \left[\int_{\Sigma} \left[|\nabla^2 \vec{v}|_{g_{\vec{\Phi}}}^2 + |\nabla \vec{v}|_{g_{\vec{\Phi}}}^2 + |\vec{v}|^2 \right]^{q/2} dvol_{g_{\vec{\Phi}}} \right]^{1/q} + \||\nabla \vec{v}|_{g_{\vec{\Phi}}} \|_{L^{\infty}(\Sigma)}$$

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Proposition $\|\cdot\|_{\vec{\Phi}}$ define a C^2 -Finsler struct. on \mathcal{M} .

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Theorem [Palais 1970] Let $(\mathcal{M}, \|\cdot\|)$ be a Finsler Manifold. Define on $\mathcal{M} \times \mathcal{M}$

$$d(p,q) := \inf_{\omega \in \Omega_{p,q}} \int_0^1 \left\| \frac{d\omega}{dt} \right\|_{\omega(t)} dt$$

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Corollary Let $(\mathcal{M}, \|\cdot\|)$ be a Finsler Manifold then \mathcal{M} is paracompact.

Completeness of the Palais Distance.

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Proposition Let q > 2 and let \mathcal{M} be the normal² Banach manifold

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is complete for the Palais distance.

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"Proof" Use that **Finsler Manifolds** are **Paracompact** and "glue together" local pseudo-gradients constructed by local trivializations with an ad-hoc partition of unity.

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Then the Dirichlet Energy satisfies the Palais Smale condition for every level set.

Admissible families

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$$\mathcal{A} := \left\{ \vec{\Phi} \in C^0([0,1], W^{2,q}_{imm}(S^2, \mathbb{R}^3)) ; \ \vec{\Phi}(0, \cdot) = \vec{\Phi}(1, \cdot) \quad \text{ and } [\vec{\Phi}] = c \right\}$$

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Assume $(PS)_{\beta}$ for the level set β . Then there exists $u \in \mathcal{M}$ s.t.

$$\begin{cases} DE_u = 0\\ E(u) = \beta \end{cases}$$

Proof

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Proof By contradiction.

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where $\operatorname{supp}(\eta) \subset [\beta - \varepsilon_0, \beta + \varepsilon]$ and $\eta \equiv 1$ on $[\beta - \varepsilon_0/2, \beta + \varepsilon_0/2]$.

$$d(\phi_{t_1}(u), \phi_{t_2}(u)) \leq 2 |t_2 - t_1|^{1/2} [E(\phi_{t_1}(u)) - E(\phi_{t_2}(u))]^{1/2}$$

If $t^u_{max} < +\infty$ then **Completeness** of $(\mathcal{M}, d) \Rightarrow$

 $\lim_{t \to t_{max}^{u}} \phi_t(u) \in \mathcal{M}^* \quad \text{Impossible } ! \Rightarrow \forall \ t \in \mathbb{R}_+ \quad \forall \ A \in \mathcal{A} \ \phi_t(A) \in \mathcal{A}$

Take
$$A \in \mathcal{A}$$
 s.t. $\max_{u \in A} E(u) < \beta + \varepsilon_0/2$.

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 $\exists \ \delta > 0 \ , \exists \ \epsilon > 0 \ \ \beta - \varepsilon < E(u) < \beta + \varepsilon \implies \|DE_u\|_u \ge \delta \quad .$

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Palais Theorem

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Palais Theorem \Rightarrow

$$W_{\vec{\sigma}_0} = \inf_{\vec{\sigma} \in \Omega_{\vec{\sigma}_0} \cap \Lambda} \max_{t \in [0,1]} E(\vec{\sigma}(t, \cdot)) > 0$$

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This gives a new proof of Birkhoff existence result.