Minmax Methods for Geodesics and Minimal Surfaces

Tristan Rivière

ETH Zürich

Lecture 2 : Palais Deformation Theory

in ∞ Dimensional Spaces.

Banach Manifolds

Definition *A* C^p **Banach Manifold** $\mathcal{M}$ *for* $p \in \mathbb{N} \cup \{\infty\}$ *is an Hausdorff topological space*

Banach Manifolds

Definition *A* C^p **Banach Manifold** $\mathcal{M}$ *for* $p \in \mathbb{N} \cup \{\infty\}$ *is an Hausdorff topological space together with a covering by open sets* $(U_i)_{i \in I}$,

Banach Manifolds

Definition *A* C^p **Banach Manifold** $\mathcal{M}$ *for* $p \in \mathbb{N} \cup \{\infty\}$ *is an Hausdorff topological space together with a covering by open sets* $(U_i)_{i \in I}$*, a family of Banach vector spaces* $(E_i)_{i \in I}$

Banach Manifolds

Definition *A* C^p **Banach Manifold** $\mathcal{M}$ *for* $p \in \mathbb{N} \cup \{\infty\}$ *is an Hausdorff topological space together with a covering by open sets* $(U_i)_{i \in I}$*, a family of Banach vector spaces* $(E_i)_{i \in I}$ *and a family of continuous mappings* $(\varphi_i)_{i \in I}$ *from* U_i *inton* E_i

Banach Manifolds

Definition *A* C^p **Banach Manifold** $\mathcal{M}$ *for* $p \in \mathbb{N} \cup \{\infty\}$ *is an Hausdorff topological space together with a covering by open sets* $(U_i)_{i\in I}$, *a family of Banach vector spaces* $(E_i)_{i\in I}$ *and a family of continuous mappings* $(\varphi_i)_{i\in I}$ *from* U_i *inton* E_i *such that*

i) *for every* $i \in I$

$$\varphi_i \; U_i \longrightarrow \varphi_i(U_i) \qquad \textit{is an homeomorphism}$$

Banach Manifolds

Definition *A* C^p **Banach Manifold** $\mathcal{M}$ *for* $p \in \mathbb{N} \cup \{\infty\}$ *is an Hausdorff topological space together with a covering by open sets* $(U_i)_{i\in I}$, *a family of Banach vector spaces* $(E_i)_{i\in I}$ *and a family of continuous mappings* $(\varphi_i)_{i\in I}$ *from* U_i *inton* E_i *such that*

i) *for every* $i \in I$

$$\varphi_i \;\; U_i \;\longrightarrow\; \varphi_i(U_i) \qquad \text{is an homeomorphism}$$

ii) *for every pair of indices* $i \neq j$ *in* I

$$\varphi_j \circ \varphi_i^{-1} \;:\; \varphi_i(U_i \cap U_j) \subset E_i \;\longrightarrow\; \varphi_j(U_i \cap U_j) \subset E_j$$

is a C^p *diffeomorphism*

□

Banach Manifolds

Definition *A* C^p **Banach Manifold** $\mathcal{M}$ *for* $p \in \mathbb{N} \cup \{\infty\}$ *is an Hausdorff topological space together with a covering by open sets* $(U_i)_{i \in I}$*, a family of Banach vector spaces* $(E_i)_{i \in I}$ *and a family of continuous mappings* $(\varphi_i)_{i \in I}$ *from* U_i *inton* E_i *such that*

i) *for every* $i \in I$

$$\varphi_i \ U_i \longrightarrow \varphi_i(U_i) \qquad \textit{is an homeomorphism}$$

ii) *for every pair of indices* $i \neq j$ *in* I

$$\varphi_j \circ \varphi_i^{-1} \ : \ \varphi_i(U_i \cap U_j) \subset E_i \longrightarrow \varphi_j(U_i \cap U_j) \subset E_j$$

is a C^p *diffeomorphism*

□

Example : Let $l\, p > k$

$$\mathcal{M} := W^{l,p}(\Sigma^k, N^n) := \left\{ \vec{u} \in W^{l,p}(\Sigma^k, \mathbb{R}^m) \ ; \ \vec{u}(x) \in N^n \ \text{ a.e. } x \in \Sigma^k \right\}$$

Paracompact Banach Manifolds

Definition *A topological Hausdorff space is called* **paracompact** *if every open covering admits a locally finite*[1] *open refinement.* □

[1] *locally finite* means that any point posses a neighborhood which intersects only finitely many open sets of the sub-covering

Paracompact Banach Manifolds

Definition *A topological Hausdorff space is called* **paracompact** *if every open covering admits a locally finite*[1] *open refinement.* □

Theorem **[Stone 1948]** *Every* **metric** *space is* **paracompact***.* □

[1] *locally finite* means that any point posses a neighborhood which intersects only finitely many open sets of the sub-covering

Paracompact Banach Manifolds

Definition *A topological Hausdorff space is called* **paracompact** *if every open covering admits a locally finite*[1] *open refinement.* □

Theorem **[Stone 1948]** *Every* **metric** *space is* **paracompact**. □

Definition *A topological space is called* **normal** *if any pair of disjoint closed sets have disjoint open neighborhoods.* □

[1] *locally finite* means that any point posses a neighborhood which intersects only finitely many open sets of the sub-covering

Paracompact Banach Manifolds

Definition *A topological Hausdorff space is called* **paracompact** *if every open covering admits a locally finite*[1] *open refinement.* □

Theorem **[Stone 1948]** *Every* **metric** *space is* **paracompact**. □

Definition *A topological space is called* **normal** *if any pair of disjoint closed sets have disjoint open neighborhoods.* □

Proposition *Every* **Hausdorff paracompact** *space is* **normal**. □

[1] *locally finite* means that any point posses a neighborhood which intersects only finitely many open sets of the sub-covering

Paracompact Banach Manifolds

Definition *A topological Hausdorff space is called* **paracompact** *if every open covering admits a locally finite*[1] *open refinement.* □

Theorem **[Stone 1948]** *Every* **metric** *space is* **paracompact**. □

Definition *A topological space is called* **normal** *if any pair of disjoint closed sets have disjoint open neighborhoods.* □

Proposition *Every* **Hausdorff paracompact** *space is* **normal**. □

Proof : https://topospaces.subwiki.org/wiki/

[1] *locally finite* means that any point posses a neighborhood which intersects only finitely many open sets of the sub-covering

Paracompact Banach Manifolds

Definition *A topological Hausdorff space is called* **paracompact** *if every open covering admits a locally finite*[1] *open refinement.* □

Theorem **[Stone 1948]** *Every* **metric** *space is* **paracompact***.* □

Definition *A topological space is called* **normal** *if any pair of disjoint closed sets have disjoint open neighborhoods.* □

Proposition *Every* **Hausdorff paracompact** *space is* **normal***.* □

Proof : https://topospaces.subwiki.org/wiki/

Warning ! $\mathcal{M}$ **Banach Paracompact Manifold**, (ϕ, U) a chart s.t.

$$\phi \ : \ U \longrightarrow \phi(U) = (E, \|\cdot\|) \qquad \text{homeomorphism}$$

[1] *locally finite* means that any point posses a neighborhood which intersects only finitely many open sets of the sub-covering

Paracompact Banach Manifolds

Definition *A topological Hausdorff space is called* **paracompact** *if every open covering admits a locally finite*[1] *open refinement.* □

Theorem **[Stone 1948]** *Every* **metric** *space is* **paracompact**. □

Definition *A topological space is called* **normal** *if any pair of disjoint closed sets have disjoint open neighborhoods.* □

Proposition *Every* **Hausdorff paracompact** *space is* **normal**. □

Proof : https://topospaces.subwiki.org/wiki/

Warning ! $\mathcal{M}$ **Banach Paracompact Manifold**, (ϕ, U) a chart s.t.

$$\phi \ : \ U \longrightarrow \phi(U) = (E, \|\cdot\|) \qquad \text{homeomorphism}$$

then $\phi^{-1}(\overline{B_r(x)})$ might not be closed in $\mathcal{M}$.

[1] *locally finite* means that any point posses a neighborhood which intersects only finitely many open sets of the sub-covering

Partition of Unity on Paracompact Banach Manifolds

Partition of Unity on Paracompact Banach Manifolds

Proposition *Let* $(\mathcal{O}_\alpha)_{\alpha \in A}$ *be an arbitrary covering of a* C^1 *paracompact Banach manifold* $\mathcal{M}$*.*

Partition of Unity on Paracompact Banach Manifolds

Proposition *Let* $(\mathcal{O}_\alpha)_{\alpha \in A}$ *be an arbitrary covering of a* C^1 *paracompact Banach manifold* $\mathcal{M}$. *Then there exists a locally* *lipschitz* *partition of unity subordinated to* $(\mathcal{O}_\alpha)_{\alpha \in A}$

Partition of Unity on Paracompact Banach Manifolds

Proposition *Let* $(\mathcal{O}_\alpha)_{\alpha\in A}$ *be an arbitrary covering of a* C^1 *paracompact Banach manifold* $\mathcal{M}$*. Then there exists a locally* *lipschitz* *partition of unity subordinated to* $(\mathcal{O}_\alpha)_{\alpha\in A}$*, i.e. there exists* $(\phi_\alpha)_{\alpha\in A}$ *where* ϕ_α *is locally lipschitz in* $\mathcal{M}$ *and such that*

Partition of Unity on Paracompact Banach Manifolds

Proposition *Let* $(\mathcal{O}_\alpha)_{\alpha \in A}$ *be an arbitrary covering of a* C^1 *paracompact Banach manifold* $\mathcal{M}$. *Then there exists a locally* *lipschitz* *partition of unity subordinated to* $(\mathcal{O}_\alpha)_{\alpha \in A}$, *i.e. there exists* $(\phi_\alpha)_{\alpha \in A}$ *where* ϕ_α *is locally lipschitz in* $\mathcal{M}$ *and such that*

i)

$$Supp(\phi_\alpha) \subset \mathcal{O}_\alpha$$

Partition of Unity on Paracompact Banach Manifolds

Proposition *Let* $(\mathcal{O}_\alpha)_{\alpha\in A}$ *be an arbitrary covering of a* C^1 *paracompact Banach manifold* $\mathcal{M}$. *Then there exists a locally* *lipschitz* *partition of unity subordinated to* $(\mathcal{O}_\alpha)_{\alpha\in A}$, *i.e. there exists* $(\phi_\alpha)_{\alpha\in A}$ *where* ϕ_α *is locally lipschitz in* $\mathcal{M}$ *and such that*

i)

$$Supp(\phi_\alpha) \subset \mathcal{O}_\alpha$$

ii)

$$\phi_\alpha \geq 0$$

Partition of Unity on Paracompact Banach Manifolds

Proposition *Let* $(\mathcal{O}_\alpha)_{\alpha \in A}$ *be an arbitrary covering of a* C^1 *paracompact Banach manifold* $\mathcal{M}$*. Then there exists a locally* *lipschitz* *partition of unity subordinated to* $(\mathcal{O}_\alpha)_{\alpha \in A}$*, i.e. there exists* $(\phi_\alpha)_{\alpha \in A}$ *where* ϕ_α *is locally lipschitz in* $\mathcal{M}$ *and such that*

i)
$$Supp(\phi_\alpha) \subset \mathcal{O}_\alpha$$

ii)
$$\phi_\alpha \geq 0$$

iii)
$$\sum_{\alpha \in A} \phi_\alpha \equiv 1$$

Partition of Unity on Paracompact Banach Manifolds

Proposition *Let $(\mathcal{O}_\alpha)_{\alpha \in A}$ be an arbitrary covering of a C^1 paracompact Banach manifold $\mathcal{M}$. Then there exists a locally lipschitz partition of unity subordinated to $(\mathcal{O}_\alpha)_{\alpha \in A}$, i.e. there exists $(\phi_\alpha)_{\alpha \in A}$ where ϕ_α is locally lipschitz in $\mathcal{M}$ and such that*

i)
$$Supp(\phi_\alpha) \subset \mathcal{O}_\alpha$$

ii)
$$\phi_\alpha \geq 0$$

iii)
$$\sum_{\alpha \in A} \phi_\alpha \equiv 1$$

where the sum is locally finite.

Banach Space Bundles

Banach Space Bundles

Definition *A Banach manifold $\mathcal{V}$ is called C^p-* **Banach Space Bundle** *over another Banach manifold $\mathcal{M}$*

Banach Space Bundles

Definition *A Banach manifold $\mathcal{V}$ is called C^p-* **Banach Space Bundle** *over another Banach manifold $\mathcal{M}$ if there exists a Banach Space E,*

Banach Space Bundles

Definition *A Banach manifold* $\mathcal{V}$ *is called* C^p- **Banach Space Bundle** *over another Banach manifold* $\mathcal{M}$ *if there exists a Banach Space* E, *a submersion* π *from* $\mathcal{V}$ *into* $\mathcal{M}$,

Banach Space Bundles

Definition *A Banach manifold $\mathcal{V}$ is called C^p-* **Banach Space Bundle** *over another Banach manifold $\mathcal{M}$ if there exists a Banach Space E, a submersion π from $\mathcal{V}$ into $\mathcal{M}$, a covering $(U_i)_{i \in I}$ of $\mathcal{M}$*

Banach Space Bundles

Definition *A Banach manifold* $\mathcal{V}$ *is called* C^p- **Banach Space Bundle** *over another Banach manifold* $\mathcal{M}$ *if there exists a Banach Space* E*, a submersion* π *from* $\mathcal{V}$ *into* $\mathcal{M}$*, a covering* $(U_i)_{i \in I}$ *of* $\mathcal{M}$ *and a family of homeomorphism from* $\pi^{-1}U_i$ *into* $U_i \times E$

Banach Space Bundles

Definition *A Banach manifold $\mathcal{V}$ is called C^p-* **Banach Space Bundle** *over another Banach manifold $\mathcal{M}$ if there exists a Banach Space E, a submersion π from $\mathcal{V}$ into $\mathcal{M}$, a covering $(U_i)_{i \in I}$ of $\mathcal{M}$ and a family of homeomorphism from $\pi^{-1}U_i$ into $U_i \times E$such that the following diagram commutes*

$$\begin{array}{ccc} \pi^{-1}U_i & \xrightarrow{\tau_i} & U_i \times E \\ & \pi \searrow & \downarrow \sigma \\ & & U_i \end{array}$$

Banach Space Bundles

Definition *A Banach manifold $\mathcal{V}$ is called C^p-* **Banach Space Bundle** *over another Banach manifold $\mathcal{M}$ if there exists a Banach Space E, a submersion π from $\mathcal{V}$ into $\mathcal{M}$, a covering $(U_i)_{i\in I}$ of $\mathcal{M}$ and a family of homeomorphism from $\pi^{-1}U_i$ into $U_i \times E$such that the following diagram commutes*

$$\begin{array}{ccc} \pi^{-1}U_i & \xrightarrow{\tau_i} & U_i \times E \\ & \pi \searrow & \downarrow \sigma \\ & & U_i \end{array}$$

where σ is the canonical projection from $U_i \times E$ onto U_i.

Banach Space Bundles

Definition *A Banach manifold $\mathcal{V}$ is called C^p-* **Banach Space Bundle** *over another Banach manifold $\mathcal{M}$ if there exists a Banach Space E, a submersion π from $\mathcal{V}$ into $\mathcal{M}$, a covering $(U_i)_{i\in I}$ of $\mathcal{M}$ and a family of homeomorphism from $\pi^{-1}U_i$ into $U_i \times E$such that the following diagram commutes*

$$\begin{array}{ccc} \pi^{-1}U_i & \xrightarrow{\tau_i} & U_i \times E \\ & \pi\searrow & \downarrow \sigma \\ & & U_i \end{array}$$

where σ is the canonical projection from $U_i \times E$ onto U_i. The restriction of τ_i on each fiber $\mathcal{V}_x := \pi^{-1}(\{x\})$ for $x \in U_i$ realizes a continuous isomorphism onto E.

Banach Space Bundles

Definition *A Banach manifold $\mathcal{V}$ is called C^p-* **Banach Space Bundle** *over another Banach manifold $\mathcal{M}$ if there exists a Banach Space E, a submersion π from $\mathcal{V}$ into $\mathcal{M}$, a covering $(U_i)_{i\in I}$ of $\mathcal{M}$ and a family of homeomorphism from $\pi^{-1}U_i$ into $U_i \times E$such that the following diagram commutes*

$$\begin{array}{ccc} \pi^{-1}U_i & \xrightarrow{\tau_i} & U_i \times E \\ & \pi \searrow & \downarrow \sigma \\ & & U_i \end{array}$$

where σ is the canonical projection from $U_i \times E$ onto U_i. The restriction of τ_i on each fiber $\mathcal{V}_x := \pi^{-1}(\{x\})$ for $x \in U_i$ realizes a continuous isomorphism onto E. Moreover the map

$$x \in U_i \cap U_j \longrightarrow \tau_i \circ \tau_j^{-1}\Big|_{\pi^{-1}(x)} \in \mathcal{L}(E, E)$$

is C^p. □

Finsler Structures on Banach Bundles.

Definition *Let $\mathcal{M}$ be a normal Banach manifold and let $\mathcal{V}$ be a Banach Space Bundle over $\mathcal{M}$.*

Finsler Structures on Banach Bundles.

Definition *Let $\mathcal{M}$ be a normal Banach manifold and let $\mathcal{V}$ be a Banach Space Bundle over $\mathcal{M}$. A* **Finsler structure** *on $\mathcal{V}$ is a continuous function*

$$\|\cdot\| \ : \ \mathcal{V} \ \longrightarrow \ \mathbb{R}$$

Finsler Structures on Banach Bundles.

Definition *Let $\mathcal{M}$ be a normal Banach manifold and let $\mathcal{V}$ be a Banach Space Bundle over $\mathcal{M}$. A* **Finsler structure** *on $\mathcal{V}$ is a continuous function*

$$\|\cdot\| \ : \ \mathcal{V} \ \longrightarrow \ \mathbb{R}$$

such that for any $x \in \mathcal{M}$

$$\|\cdot\|_x := \|\cdot\||_{\pi^{-1}(\{x\})} \qquad \text{is a norm on } \mathcal{V}_x \quad .$$

Finsler Structures on Banach Bundles.

Definition *Let $\mathcal{M}$ be a normal Banach manifold and let $\mathcal{V}$ be a Banach Space Bundle over $\mathcal{M}$. A* **Finsler structure** *on $\mathcal{V}$ is a continuous function*

$$\|\cdot\| \ : \ \mathcal{V} \ \longrightarrow \ \mathbb{R}$$

such that for any $x \in \mathcal{M}$

$$\|\cdot\|_x := \|\cdot\||_{\pi^{-1}(\{x\})} \qquad \text{is a norm on } \mathcal{V}_x \quad .$$

and the norms are locally uniformly comparable using any trivialization. □

Finsler Structures on Banach Bundles.

Definition *Let $\mathcal{M}$ be a normal Banach manifold and let $\mathcal{V}$ be a Banach Space Bundle over $\mathcal{M}$. A* **Finsler structure** *on $\mathcal{V}$ is a continuous function*

$$\|\cdot\| \;:\; \mathcal{V} \;\longrightarrow\; \mathbb{R}$$

such that for any $x \in \mathcal{M}$

$$\|\cdot\|_x := \|\cdot\||_{\pi^{-1}(\{x\})} \qquad \text{is a norm on } \mathcal{V}_x \quad .$$

and the norms are locally uniformly comparable using any trivialization. □

Definition *Let $\mathcal{M}$ be a* **normal** *C^p Banach manifold. $T\mathcal{M}$ equipped with a Finsler structure is called a* **Finsler Manifold**.

A Finsler Structure on Sobolev Immersions.

A Finsler Structure on Sobolev Immersions.

Let Σ^2 be a closed oriented $2-$dim manifold and N^n be a closed sub-manifold of $\mathbb{R}^m$.

A Finsler Structure on Sobolev Immersions.

Let Σ^2 be a closed oriented $2-$dim manifold and N^n be a closed sub-manifold of $\mathbb{R}^m$. Let $q > 2$

$$\mathcal{M} := W^{2,q}_{imm}(\Sigma^2, N^n)$$

$$:= \left\{ \vec{\Phi} \in W^{2,q}(\Sigma^2, N^n) \ ; \ \text{rank}\,(d\Phi_x) = 2 \quad \forall x \in \Sigma^2 \right\}$$

A Finsler Structure on Sobolev Immersions.

Let Σ^2 be a closed oriented 2−dim manifold and N^n be a closed sub-manifold of $\mathbb{R}^m$. Let $q > 2$

$$\mathcal{M} := W^{2,q}_{imm}(\Sigma^2, N^n)$$

$$:= \left\{ \vec{\Phi} \in W^{2,q}(\Sigma^2, N^n) \ ; \ \text{rank}\,(d\Phi_x) = 2 \quad \forall x \in \Sigma^2 \right\}$$

The tangent space to $\mathcal{M}$ at a point $\vec{\Phi}$ is

$$T_{\vec{\Phi}}\mathcal{M} = \left\{ \vec{w} \in W^{2,q}(\Sigma^2, \mathbb{R}^m) \ ; \ \vec{w}(x) \in T_{\vec{\Phi}(x)} N^n \quad \forall\, x \in \Sigma^2 \right\} \quad .$$

A Finsler Structure on Sobolev Immersions.

Let Σ^2 be a closed oriented $2-$dim manifold and N^n be a closed sub-manifold of $\mathbb{R}^m$. Let $q>2$

$$\begin{aligned}\mathcal{M} &:= W^{2,q}_{imm}(\Sigma^2,N^n)\\ &:= \left\{\vec{\Phi}\in W^{2,q}(\Sigma^2,N^n)\ ;\ \mathrm{rank}\,(d\Phi_x)=2\quad \forall x\in\Sigma^2\right\}\end{aligned}$$

The tangent space to $\mathcal{M}$ at a point $\vec{\Phi}$ is

$$T_{\vec{\Phi}}\mathcal{M}=\left\{\vec{w}\in W^{2,q}(\Sigma^2,\mathbb{R}^m)\ ;\ \vec{w}(x)\in T_{\vec{\Phi}(x)}N^n\quad \forall\, x\in\ \Sigma^2\right\}\quad .$$

We equip $T_{\vec{\Phi}}\mathcal{M}$ with the following norm

$$\|\vec{v}\|_{\vec{\Phi}}:=\left[\int_\Sigma\left[|\nabla^2\vec{v}|^2_{g_{\vec{\Phi}}}+|\nabla\vec{v}|^2_{g_{\vec{\Phi}}}+|\vec{v}|^2\right]^{q/2}\ dvol_{g_{\vec{\Phi}}}\right]^{1/q}+\|\,|\nabla\vec{v}|_{g_{\vec{\Phi}}}\,\|_{L^\infty(\Sigma)}$$

A Finsler Structure on Sobolev Immersions.

Let Σ^2 be a closed oriented $2-$dim manifold and N^n be a closed sub-manifold of $\mathbb{R}^m$. Let $q > 2$

$$\mathcal{M} := W^{2,q}_{imm}(\Sigma^2, N^n)$$

$$:= \left\{\vec{\Phi} \in W^{2,q}(\Sigma^2, N^n)\ ;\ \text{rank}\,(d\Phi_x) = 2 \quad \forall x \in \Sigma^2\right\}$$

The tangent space to $\mathcal{M}$ at a point $\vec{\Phi}$ is

$$T_{\vec{\Phi}}\mathcal{M} = \left\{\vec{w} \in W^{2,q}(\Sigma^2, \mathbb{R}^m)\ ;\ \vec{w}(x) \in T_{\vec{\Phi}(x)}N^n \quad \forall\, x \in \Sigma^2\right\} \quad .$$

We equip $T_{\vec{\Phi}}\mathcal{M}$ with the following norm

$$\|\vec{v}\|_{\vec{\Phi}} := \left[\int_\Sigma \left[|\nabla^2\vec{v}|^2_{g_{\vec{\Phi}}} + |\nabla\vec{v}|^2_{g_{\vec{\Phi}}} + |\vec{v}|^2\right]^{q/2}\ dvol_{g_{\vec{\Phi}}}\right]^{1/q} + \|\,|\nabla\vec{v}|_{g_{\vec{\Phi}}}\,\|_{L^\infty(\Sigma)}$$

Proposition $\|\cdot\|_{\vec{\Phi}}$ *define a* C^2-*Finsler struct. on* $\mathcal{M}$. □

The Palais Distance.

The Palais Distance.

Theorem **[Palais 1970]** Let $(\mathcal{M}, \|\cdot\|)$ be a **Finsler Manifold**. Define on $\mathcal{M} \times \mathcal{M}$

$$d(p,q) := \inf_{\omega \in \Omega_{p,q}} \int_0^1 \left\| \frac{d\omega}{dt} \right\|_{\omega(t)} dt$$

The Palais Distance.

Theorem **[Palais 1970]** Let $(\mathcal{M}, \|\cdot\|)$ be a **Finsler Manifold**. Define on $\mathcal{M} \times \mathcal{M}$

$$d(p,q) := \inf_{\omega \in \Omega_{p,q}} \int_0^1 \left\| \frac{d\omega}{dt} \right\|_{\omega(t)} dt$$

where

$$\Omega_{p,q} := \left\{ \omega \in C^1([0,1], \mathcal{M}) \; ; \; \omega(0) = p \quad \omega(1) = q \right\} \quad .$$

The Palais Distance.

Theorem **[Palais 1970]** Let $(\mathcal{M}, \|\cdot\|)$ be a **Finsler Manifold**. Define on $\mathcal{M} \times \mathcal{M}$

$$d(p,q) := \inf_{\omega \in \Omega_{p,q}} \int_0^1 \left\| \frac{d\omega}{dt} \right\|_{\omega(t)} \, dt$$

where

$$\Omega_{p,q} := \left\{ \omega \in C^1([0,1], \mathcal{M}) \; ; \; \omega(0) = p \quad \omega(1) = q \right\} \quad .$$

Then d defines a distance on $\mathcal{M}$ and $(\mathcal{M}, d)$ defines the same topology as the one of the Banach Manifold.

The Palais Distance.

Theorem **[Palais 1970]** Let $(\mathcal{M}, \| \cdot \|)$ be a **Finsler Manifold**. Define on $\mathcal{M} \times \mathcal{M}$

$$d(p,q) := \inf_{\omega \in \Omega_{p,q}} \int_0^1 \left\| \frac{d\omega}{dt} \right\|_{\omega(t)} \, dt$$

where

$$\Omega_{p,q} := \left\{ \omega \in C^1([0,1], \mathcal{M}) \ ; \ \omega(0) = p \quad \omega(1) = q \right\} \quad .$$

Then d defines a distance on $\mathcal{M}$ and $(\mathcal{M}, d)$ defines the same topology as the one of the Banach Manifold.

d is called **Palais distance** of the Finsler manifold $(\mathcal{M}, \| \cdot \|)$.

The Palais Distance.

Theorem **[Palais 1970]** Let $(\mathcal{M}, \|\cdot\|)$ be a **Finsler Manifold**. Define on $\mathcal{M} \times \mathcal{M}$

$$d(p,q) := \inf_{\omega \in \Omega_{p,q}} \int_0^1 \left\| \frac{d\omega}{dt} \right\|_{\omega(t)} dt$$

where

$$\Omega_{p,q} := \left\{ \omega \in C^1([0,1], \mathcal{M}) \ ; \ \omega(0) = p \quad \omega(1) = q \right\} \quad .$$

Then d defines a distance on $\mathcal{M}$ and $(\mathcal{M}, d)$ defines the same topology as the one of the Banach Manifold.

d is called **Palais distance** of the Finsler manifold $(\mathcal{M}, \|\cdot\|)$.

Corollary *Let* $(\mathcal{M}, \|\cdot\|)$ *be a* **Finsler Manifold** *then* $\mathcal{M}$ *is* **paracompact**. □

Completeness of the Palais Distance.

[2] Recall that every metric space is normal.

Completeness of the Palais Distance.

Proposition *Let $q > 2$ and let $\mathcal{M}$ be the* **normal**[2] **Banach manifold**

$$W^{2,q}_{imm}(\Sigma^2, N^n) := \left\{\vec{\Phi} \in W^{2,q}(\Sigma^2, N^n)\ ;\ rank(d\Phi_x) = 2 \quad \forall x \in \Sigma^2\right\}$$

[2]Recall that every metric space is normal.

Completeness of the Palais Distance.

Proposition *Let* $q>2$ *and let* $\mathcal{M}$ *be the* **normal**[2] **Banach manifold**

$$W^{2,q}_{imm}(\Sigma^2,N^n):=\left\{\vec{\Phi}\in W^{2,q}(\Sigma^2,N^n)\ ;\ rank(d\Phi_x)=2\quad\forall x\in\Sigma^2\right\}$$

The **Finsler Manifold** *given by*

$$\|\vec{v}\|_{\vec{\Phi}}:=\left[\int_\Sigma\left[|\nabla^2\vec{v}|^2_{g_{\vec{\Phi}}}+|\nabla\vec{v}|^2_{g_{\vec{\Phi}}}+|\vec{v}|^2\right]^{q/2}\ dvol_{g_{\vec{\Phi}}}\right]^{1/q}+\|\,|\nabla\vec{v}|_{g_{\vec{\Phi}}}\,\|_{L^\infty(\Sigma)}$$

is **complete** *for the* **Palais distance**.

[2] Recall that every metric space is normal.

Pseudo-gradients

Pseudo-gradients

Definition *Let $\mathcal{M}$ be a C^2 Finsler Manifold and E be a C^1 function on $\mathcal{M}$.*

Pseudo-gradients

Definition *Let $\mathcal{M}$ be a C^2 Finsler Manifold and E be a C^1 function on $\mathcal{M}$. Denote*

$$\mathcal{M}^* := \{u \in \mathcal{M} \quad ; \quad DE_u \neq 0\} \quad .$$

Pseudo-gradients

Definition *Let* $\mathcal{M}$ *be a* C^2 *Finsler Manifold and* E *be a* C^1 *function on* $\mathcal{M}$. *Denote*

$$\mathcal{M}^* := \{u \in \mathcal{M} \quad ; \quad DE_u \neq 0\} \quad .$$

A **pseudo-gradient** *is a Lipschitz continuous section* $X \ : \ \mathcal{M}^* \to T\mathcal{M}^*$ *such that*

Pseudo-gradients

Definition *Let* $\mathcal{M}$ *be a* C^2 *Finsler Manifold and* E *be a* C^1 *function on* $\mathcal{M}$*. Denote*

$$\mathcal{M}^* := \{u \in \mathcal{M} \quad ; \quad DE_u \neq 0\} \quad .$$

A **pseudo-gradient** *is a Lipschitz continuous section* $X \ : \ \mathcal{M}^* \to T\mathcal{M}^*$ *such that*

i)

$$\forall\, u \in \mathcal{M}^* \qquad \|X(u)\|_u < 2\ \|DE_u\|_u$$

Pseudo-gradients

Definition *Let $\mathcal{M}$ be a C^2 Finsler Manifold and E be a C^1 function on $\mathcal{M}$. Denote*

$$\mathcal{M}^* := \{u \in \mathcal{M} \quad ; \quad DE_u \neq 0\} \quad .$$

A **pseudo-gradient** *is a Lipschitz continuous section $X \; : \; \mathcal{M}^* \to T\mathcal{M}^*$ such that*

i)

$$\forall\, u \in \mathcal{M}^* \qquad \|X(u)\|_u < 2\, \|DE_u\|_u$$

ii)

$$\forall\, u \in \mathcal{M}^* \quad \|DE_u\|_u^2 < \langle X(u), DE_u \rangle_{T_u\mathcal{M}^*, T_u^*\mathcal{M}^*}$$

Pseudo-gradients

Definition *Let $\mathcal{M}$ be a C^2 Finsler Manifold and E be a C^1 function on $\mathcal{M}$. Denote*

$$\mathcal{M}^* := \{u \in \mathcal{M} \quad ; \quad DE_u \neq 0\} \quad .$$

A **pseudo-gradient** *is a Lipschitz continuous section $X \; : \; \mathcal{M}^* \to T\mathcal{M}^*$ such that*

i)

$$\forall \, u \in \mathcal{M}^* \qquad \|X(u)\|_u < 2 \; \|DE_u\|_u$$

ii)

$$\forall \, u \in \mathcal{M}^* \quad \|DE_u\|_u^2 < \langle X(u), DE_u \rangle_{T_u\mathcal{M}^*, T_u^*\mathcal{M}^*}$$

Proposition *Every C^1 function on a Finsler Manifold admits a pseudo-gradient.* □

Pseudo-gradients

Definition *Let $\mathcal{M}$ be a C^2 Finsler Manifold and E be a C^1 function on $\mathcal{M}$. Denote*

$$\mathcal{M}^* := \{u \in \mathcal{M} \quad ; \quad DE_u \neq 0\} \quad .$$

A **pseudo-gradient** *is a Lipschitz continuous section $X \; : \; \mathcal{M}^* \to T\mathcal{M}^*$ such that*

i)

$$\forall\, u \in \mathcal{M}^* \qquad \|X(u)\|_u < 2\, \|DE_u\|_u$$

ii)

$$\forall\, u \in \mathcal{M}^* \quad \|DE_u\|_u^2 < \langle X(u), DE_u \rangle_{T_u\mathcal{M}^*, T_u^*\mathcal{M}^*}$$

Proposition *Every C^1 function on a Finsler Manifold admits a pseudo-gradient.* □

"Proof" Use that **Finsler Manifolds** are **Paracompact** and "glue together" local pseudo-gradients constructed by local trivializations with an ad-hoc partition of unity.

The Palais-Smale condition : (PS)

The Palais-Smale condition : (PS)

Definition *Let E be a C^1 function on a Finsler manifold $(\mathcal{M}, \|\cdot\|)$ and $\beta \in E(\mathcal{M})$.*

The Palais-Smale condition : (PS)

Definition *Let* E *be a* C^1 *function on a Finsler manifold* $(\mathcal{M}, \|\cdot\|)$ *and* $\beta \in E(\mathcal{M})$. *One says that* E *fulfills the* **Palais-Smale condition** *at the level* β *if for any sequence* u_n *satisfying*

$$E(u_n) \longrightarrow \beta \quad \text{and} \quad \|DE_{u_n}\|_{u_n} \longrightarrow 0 \quad ,$$

The Palais-Smale condition : (PS)

Definition *Let E be a C^1 function on a Finsler manifold $(\mathcal{M}, \|\cdot\|)$ and $\beta \in E(\mathcal{M})$. One says that E fulfills the* **Palais-Smale condition** *at the level β if for any sequence u_n satisfying*

$$E(u_n) \longrightarrow \beta \quad \text{and} \quad \|DE_{u_n}\|_{u_n} \longrightarrow 0 \quad ,$$

then there exists a subsequence $u_{n'}$ and $u_\infty \in \mathcal{M}$ such that

$$d_{\mathbf{P}}(u_{n'}, u_\infty) \longrightarrow 0 \quad .$$

The Palais-Smale condition : (PS)

Definition *Let E be a C^1 function on a Finsler manifold $(\mathcal{M}, \|\cdot\|)$ and $\beta \in E(\mathcal{M})$. One says that E fulfills the* **Palais-Smale condition** *at the level β if for any sequence u_n satisfying*

$$E(u_n) \longrightarrow \beta \quad \text{and} \quad \|DE_{u_n}\|_{u_n} \longrightarrow 0 \quad ,$$

then there exists a subsequence $u_{n'}$ and $u_\infty \in \mathcal{M}$ such that

$$d_{\mathbf{P}}(u_{n'}, u_\infty) \longrightarrow 0 \quad .$$

and hence $E(u_\infty) = \beta$ and $DE_{u_\infty} = 0$. □

The Palais-Smale condition : (PS)

Definition *Let E be a C^1 function on a Finsler manifold $(\mathcal{M}, \|\cdot\|)$ and $\beta \in E(\mathcal{M})$. One says that E fulfills the* **Palais-Smale condition** *at the level β if for any sequence u_n satisfying*

$$E(u_n) \longrightarrow \beta \quad \text{and} \quad \|DE_{u_n}\|_{u_n} \longrightarrow 0 \quad ,$$

then there exists a subsequence $u_{n'}$ and $u_\infty \in \mathcal{M}$ such that

$$d_{\mathbf{P}}(u_{n'}, u_\infty) \longrightarrow 0 \quad .$$

and hence $E(u_\infty) = \beta$ and $DE_{u_\infty} = 0$. □

Example Let $\mathcal{M}$ be $W^{1,2}(S^1, N^n)$ for the Finsler structure given by

$$\forall\ \vec{w} \in \Gamma_{W^{1,2}}(\vec{u}^{-1}TN^n) \qquad \|\vec{w}\|_{\vec{u}} := \|\vec{w}\|_{W^{1,2}(S^1)}$$

The Palais-Smale condition : (PS)

Definition *Let E be a C^1 function on a Finsler manifold $(\mathcal{M}, \|\cdot\|)$ and $\beta \in E(\mathcal{M})$. One says that E fulfills the* **Palais-Smale condition** *at the level β if for any sequence u_n satisfying*

$$E(u_n) \longrightarrow \beta \quad \text{and} \quad \|DE_{u_n}\|_{u_n} \longrightarrow 0 \quad ,$$

then there exists a subsequence $u_{n'}$ and $u_\infty \in \mathcal{M}$ such that

$$d_{\mathbf{P}}(u_{n'}, u_\infty) \longrightarrow 0 \quad .$$

and hence $E(u_\infty) = \beta$ and $DE_{u_\infty} = 0$. □

Example Let $\mathcal{M}$ be $W^{1,2}(S^1, N^n)$ for the Finsler structure given by

$$\forall\ \vec{w} \in \Gamma_{W^{1,2}}(\vec{u}^{-1}TN^n) \qquad \|\vec{w}\|_{\vec{u}} := \|\vec{w}\|_{W^{1,2}(S^1)}$$

Then the Dirichlet Energy satisfies the Palais Smale condition for every level set. □

Admissible families

Definition *A family of closed subsets* $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ *of a Banach manifold* $\mathcal{M}$ *is called* **admissible family**

Admissible families

Definition *A family of closed subsets* $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ *of a Banach manifold* $\mathcal{M}$ *is called* **admissible family** *if for every homeomorphism* Ξ *of* $\mathcal{M}$ *isotopic to the identity we have*

$$\forall\, A \in \mathcal{A} \qquad \Xi(A) \in \mathcal{A}$$

□

Admissible families

Definition *A family of closed subsets* $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ *of a Banach manifold* $\mathcal{M}$ *is called* **admissible family** *if for every homeomorphism* Ξ *of* $\mathcal{M}$ *isotopic to the identity we have*

$$\forall\, A \in \mathcal{A} \qquad \Xi(A) \in \mathcal{A}$$

$\square$

Example $\mathcal{M} := W^{2,q}_{imm}(S^2, \mathbb{R}^3)$.

Admissible families

Definition *A family of closed subsets* $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ *of a Banach manifold* $\mathcal{M}$ *is called* **admissible family** *if for every homeomorphism* Ξ *of* $\mathcal{M}$ *isotopic to the identity we have*

$$\forall\, A \in \mathcal{A} \qquad \Xi(A) \in \mathcal{A}$$

□

Example $\mathcal{M} := W^{2,q}_{imm}(S^2, \mathbb{R}^3)$. Let $c \in \pi_1(\mathrm{Imm}(S^2, \mathbb{R}^3)) = \mathbb{Z}_2 \times \mathbb{Z}$

Admissible families

Definition *A family of closed subsets $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ of a Banach manifold $\mathcal{M}$ is called* **admissible family** *if for every homeomorphism Ξ of $\mathcal{M}$ isotopic to the identity we have*

$$\forall\, A \in \mathcal{A} \qquad \Xi(A) \in \mathcal{A}$$

□

Example $\mathcal{M} := W^{2,q}_{imm}(S^2, \mathbb{R}^3)$. Let $c \in \pi_1(\mathrm{Imm}(S^2, \mathbb{R}^3)) = \mathbb{Z}_2 \times \mathbb{Z}$ then

$$\mathcal{A} := \left\{ \vec{\Phi} \in C^0([0,1], W^{2,q}_{imm}(S^2, \mathbb{R}^3)) \; ; \; \vec{\Phi}(0,\cdot) = \vec{\Phi}(1,\cdot) \quad \text{and } [\vec{\Phi}] = c \right\}$$

is admissible

Palais Min-Max Principle

Palais Min-Max Principle

Theorem[Palais 1970] Let $(\mathcal{M}, \|\cdot\|)$ be a $C^{1,1}-$Finsler manifold.

Palais Min-Max Principle

Theorem[Palais 1970] Let $(\mathcal{M}, \|\cdot\|)$ be a $C^{1,1}-$Finsler manifold. Assume $\mathcal{M}$ is complete for $d_{\mathbf{P}}$

Palais Min-Max Principle

Theorem[Palais 1970] Let $(\mathcal{M}, \|\cdot\|)$ be a $C^{1,1}-$Finsler manifold. Assume $\mathcal{M}$ is complete for $d_{\mathbf{P}}$ and let $E \in C^1(\mathcal{M})$.

Palais Min-Max Principle

Theorem[Palais 1970] Let $(\mathcal{M}, \|\cdot\|)$ be a $C^{1,1}-$Finsler manifold. Assume $\mathcal{M}$ is complete for $d_{\mathbf{P}}$ and let $E \in C^1(\mathcal{M})$. Let $\mathcal{A}$ admissible.

Palais Min-Max Principle

Theorem[Palais 1970] Let $(\mathcal{M}, \|\cdot\|)$ be a $C^{1,1}-$Finsler manifold. Assume $\mathcal{M}$ is complete for $d_{\mathbf{P}}$ and let $E \in C^1(\mathcal{M})$. Let $\mathcal{A}$ admissible. Let

$$\beta := \inf_{A \in \mathcal{A}} \sup_{u \in A} E(u)$$

Palais Min-Max Principle

Theorem[Palais 1970] Let $(\mathcal{M}, \|\cdot\|)$ be a $C^{1,1}-$Finsler manifold. Assume $\mathcal{M}$ is complete for $d_{\mathbf{P}}$ and let $E \in C^1(\mathcal{M})$. Let $\mathcal{A}$ admissible. Let

$$\beta := \inf_{A \in \mathcal{A}} \sup_{u \in A} E(u)$$

Assume $(PS)_\beta$ for the level set β. Then there exists $u \in \mathcal{M}$ s.t.

$$\begin{cases} DE_u = 0 \\ \\ E(u) = \beta \end{cases}$$

Proof

Proof By **contradiction**.

Proof By **contradiction**.$(\mathbf{PS})_\beta \;\Rightarrow$

Proof By **contradiction**.$(\mathbf{PS})_\beta \ \Rightarrow$

$$\exists\ \delta > 0\ , \exists\ \epsilon > 0 \quad \beta - \varepsilon < E(u) < \beta + \varepsilon \quad \Longrightarrow \quad \|DE_u\|_u \geq \delta \quad .$$

Proof By **contradiction**.$(\mathbf{PS})_\beta \;\Rightarrow$

$$\exists\ \delta > 0\ ,\exists\ \epsilon > 0 \quad \beta - \varepsilon < E(u) < \beta + \varepsilon \quad \Longrightarrow \quad \|DE_u\|_u \geq \delta \quad .$$

Let $u \in \mathcal{M}^*$ and ϕ_t

$$\begin{cases} \dfrac{d\phi_t(u)}{dt} = -\ X(\phi_t(u))\ \eta(E(\phi_t(u))) \quad \text{in } [0, t^u_{max}) \\ \\ \phi_0(u) = u \end{cases}$$

Proof By **contradiction**.$(\mathbf{PS})_\beta \;\Rightarrow$

$$\exists\ \delta > 0\ , \exists\ \epsilon > 0 \quad \beta - \varepsilon < E(u) < \beta + \varepsilon \quad \Longrightarrow \quad \|DE_u\|_u \geq \delta \quad .$$

Let $u \in \mathcal{M}^*$ and ϕ_t

$$\begin{cases} \dfrac{d\phi_t(u)}{dt} = -\ X(\phi_t(u))\ \eta(E(\phi_t(u))) \quad \text{in } [0, t^u_{max}) \\ \\ \phi_0(u) = u \end{cases}$$

where $\mathrm{supp}(\eta) \subset [\beta - \varepsilon_0, \beta + \varepsilon]$ and $\eta \equiv 1$ on $[\beta - \varepsilon_0/2, \beta + \varepsilon_0/2]$.

Proof By **contradiction**.$(\mathbf{PS})_\beta \;\Rightarrow$

$$\exists\ \delta > 0\ , \exists\ \epsilon > 0 \quad \beta - \varepsilon < E(u) < \beta + \varepsilon \quad \Longrightarrow \quad \|DE_u\|_u \geq \delta \quad .$$

Let $u \in \mathcal{M}^*$ and ϕ_t

$$\begin{cases} \dfrac{d\phi_t(u)}{dt} = -\ X(\phi_t(u))\ \eta(E(\phi_t(u))) \quad \text{in } [0, t^u_{max}) \\ \\ \phi_0(u) = u \end{cases}$$

where $\text{supp}(\eta) \subset [\beta - \varepsilon_0, \beta + \varepsilon]$ and $\eta \equiv 1$ on $[\beta - \varepsilon_0/2, \beta + \varepsilon_0/2]$.

$$d(\phi_{t_1}(u), \phi_{t_2}(u)) \leq\ 2\,|t_2 - t_1|^{1/2}\ [E(\phi_{t_1}(u)) - E(\phi_{t_2}(u))]^{1/2}$$

Proof By **contradiction**.$(\mathbf{PS})_\beta \;\Rightarrow$

$$\exists\ \delta > 0\ , \exists\ \epsilon > 0 \quad \beta - \varepsilon < E(u) < \beta + \varepsilon \quad \Longrightarrow \quad \|DE_u\|_u \geq \delta \quad .$$

Let $u \in \mathcal{M}^*$ and ϕ_t

$$\begin{cases} \dfrac{d\phi_t(u)}{dt} = -\ X(\phi_t(u))\ \eta(E(\phi_t(u))) \quad \text{in } [0, t^u_{max}) \\ \\ \phi_0(u) = u \end{cases}$$

where $\text{supp}(\eta) \subset [\beta - \varepsilon_0, \beta + \varepsilon]$ and $\eta \equiv 1$ on $[\beta - \varepsilon_0/2, \beta + \varepsilon_0/2]$.

$$d(\phi_{t_1}(u), \phi_{t_2}(u)) \leq\ 2\,|t_2 - t_1|^{1/2}\ [E(\phi_{t_1}(u)) - E(\phi_{t_2}(u))]^{1/2}$$

If $t^u_{max} < +\infty$

Proof By **contradiction**.$(\mathbf{PS})_\beta \;\Rightarrow$

$$\exists\ \delta > 0\ , \exists\ \epsilon > 0 \quad \beta - \varepsilon < E(u) < \beta + \varepsilon \quad \Longrightarrow \quad \|DE_u\|_u \geq \delta \quad .$$

Let $u \in \mathcal{M}^*$ and ϕ_t

$$\begin{cases} \dfrac{d\phi_t(u)}{dt} = -\ X(\phi_t(u))\ \eta(E(\phi_t(u))) \quad \text{in } [0, t^u_{max}) \\ \\ \phi_0(u) = u \end{cases}$$

where $\mathrm{supp}(\eta) \subset [\beta - \varepsilon_0, \beta + \varepsilon]$ and $\eta \equiv 1$ on $[\beta - \varepsilon_0/2, \beta + \varepsilon_0/2]$.

$$d(\phi_{t_1}(u), \phi_{t_2}(u)) \leq\ 2\,|t_2 - t_1|^{1/2}\ [E(\phi_{t_1}(u)) - E(\phi_{t_2}(u))]^{1/2}$$

If $t^u_{max} < +\infty$ then **Completeness** of $(\mathcal{M}, d)$

Proof By **contradiction**.$(\mathbf{PS})_\beta \;\Rightarrow$

$$\exists\, \delta > 0 \,, \exists\, \epsilon > 0 \quad \beta - \varepsilon < E(u) < \beta + \varepsilon \quad \Longrightarrow \quad \|DE_u\|_u \geq \delta \quad .$$

Let $u \in \mathcal{M}^*$ and ϕ_t

$$\begin{cases} \dfrac{d\phi_t(u)}{dt} = -\, X(\phi_t(u))\, \eta(E(\phi_t(u))) \quad \text{in } [0, t^u_{max}) \\ \\ \phi_0(u) = u \end{cases}$$

where $\mathrm{supp}(\eta) \subset [\beta - \varepsilon_0, \beta + \varepsilon]$ and $\eta \equiv 1$ on $[\beta - \varepsilon_0/2, \beta + \varepsilon_0/2]$.

$$d(\phi_{t_1}(u), \phi_{t_2}(u)) \leq \; 2\, |t_2 - t_1|^{1/2} \; [E(\phi_{t_1}(u)) - E(\phi_{t_2}(u))]^{1/2}$$

If $t^u_{max} < +\infty$ then **Completeness** of $(\mathcal{M}, d) \Rightarrow$

$$\lim_{t \to t^u_{max}} \phi_t(u) \in \mathcal{M}^*$$

Proof By **contradiction**.$(\mathbf{PS})_\beta \ \Rightarrow$

$$\exists\ \delta > 0\ , \exists\ \epsilon > 0 \quad \beta - \varepsilon < E(u) < \beta + \varepsilon \quad \Longrightarrow \quad \|DE_u\|_u \geq \delta \quad .$$

Let $u \in \mathcal{M}^*$ and ϕ_t

$$\begin{cases} \dfrac{d\phi_t(u)}{dt} = -\ X(\phi_t(u))\ \eta(E(\phi_t(u))) \quad \text{in } [0, t^u_{max}) \\ \\ \phi_0(u) = u \end{cases}$$

where $\mathrm{supp}(\eta) \subset [\beta - \varepsilon_0, \beta + \varepsilon]$ and $\eta \equiv 1$ on $[\beta - \varepsilon_0/2, \beta + \varepsilon_0/2]$.

$$d(\phi_{t_1}(u), \phi_{t_2}(u)) \leq\ 2\,|t_2 - t_1|^{1/2}\ [E(\phi_{t_1}(u)) - E(\phi_{t_2}(u))]^{1/2}$$

If $t^u_{max} < +\infty$ then **Completeness** of $(\mathcal{M}, d) \Rightarrow$

$\lim_{t \to t^u_{max}} \phi_t(u) \in \mathcal{M}^*$ Impossible !

Proof By **contradiction**.$(\mathbf{PS})_\beta \;\Rightarrow$

$$\exists\ \delta > 0\ ,\exists\ \epsilon > 0 \quad \beta - \varepsilon < E(u) < \beta + \varepsilon \quad \Longrightarrow \quad \|DE_u\|_u \geq \delta \quad .$$

Let $u \in \mathcal{M}^*$ and ϕ_t

$$\left\{ \begin{array}{l} \dfrac{d\phi_t(u)}{dt} = -\ X(\phi_t(u))\ \eta(E(\phi_t(u))) \quad \text{in } [0, t^u_{max}) \\[2ex] \phi_0(u) = u \end{array} \right.$$

where $\mathrm{supp}(\eta) \subset [\beta - \varepsilon_0, \beta + \varepsilon]$ and $\eta \equiv 1$ on $[\beta - \varepsilon_0/2, \beta + \varepsilon_0/2]$.

$$d(\phi_{t_1}(u), \phi_{t_2}(u)) \leq\ 2\,|t_2 - t_1|^{1/2}\ [E(\phi_{t_1}(u)) - E(\phi_{t_2}(u))]^{1/2}$$

If $t^u_{max} < +\infty$ then **Completeness** of $(\mathcal{M}, d) \Rightarrow$

$\lim\limits_{t \to t^u_{max}} \phi_t(u) \in \mathcal{M}^*$ Impossible ! $\Rightarrow \forall\ t \in \mathbb{R}_+ \quad \forall\ A \in \mathcal{A}\ \phi_t(A) \in \mathcal{A}$

Proof By **contradiction**.$(\mathbf{PS})_\beta \Rightarrow$

$$\exists\ \delta > 0\ , \exists\ \epsilon > 0 \quad \beta - \varepsilon < E(u) < \beta + \varepsilon \quad \Longrightarrow \quad \|DE_u\|_u \geq \delta \quad .$$

Let $u \in \mathcal{M}^*$ and ϕ_t

$$\begin{cases} \dfrac{d\phi_t(u)}{dt} = -\ X(\phi_t(u))\ \eta(E(\phi_t(u))) & \text{in } [0, t^u_{max}) \\ \phi_0(u) = u \end{cases}$$

where $\mathrm{supp}(\eta) \subset [\beta - \varepsilon_0, \beta + \varepsilon]$ and $\eta \equiv 1$ on $[\beta - \varepsilon_0/2, \beta + \varepsilon_0/2]$.

$$d(\phi_{t_1}(u), \phi_{t_2}(u)) \leq\ 2\,|t_2 - t_1|^{1/2}\ [E(\phi_{t_1}(u)) - E(\phi_{t_2}(u))]^{1/2}$$

If $t^u_{max} < +\infty$ then **Completeness** of $(\mathcal{M}, d) \Rightarrow$

$\lim_{t \to t^u_{max}} \phi_t(u) \in \mathcal{M}^*$ Impossible ! $\Rightarrow \forall\ t \in \mathbb{R}_+ \quad \forall\ A \in \mathcal{A}\ \phi_t(A) \in \mathcal{A}$

Take $A \in \mathcal{A}$ s.t. $\max_{u \in A} E(u) < \beta + \varepsilon_0/2$.

Proof By **contradiction**.$(\mathbf{PS})_\beta \ \Rightarrow$

$$\exists\ \delta > 0\ ,\exists\ \epsilon > 0 \quad \beta - \varepsilon < E(u) < \beta + \varepsilon \quad \Longrightarrow \quad \|DE_u\|_u \geq \delta \quad .$$

Let $u \in \mathcal{M}^*$ and ϕ_t

$$\begin{cases} \dfrac{d\phi_t(u)}{dt} = -\ X(\phi_t(u))\ \eta(E(\phi_t(u))) \quad \text{in } [0, t^u_{max}) \\ \\ \phi_0(u) = u \end{cases}$$

where $\mathrm{supp}(\eta) \subset [\beta - \varepsilon_0, \beta + \varepsilon]$ and $\eta \equiv 1$ on $[\beta - \varepsilon_0/2, \beta + \varepsilon_0/2]$.

$$d(\phi_{t_1}(u), \phi_{t_2}(u)) \leq \ 2\,|t_2 - t_1|^{1/2}\ [E(\phi_{t_1}(u)) - E(\phi_{t_2}(u))]^{1/2}$$

If $t^u_{max} < +\infty$ then **Completeness** of $(\mathcal{M}, d) \Rightarrow$

$\lim_{t \to t^u_{max}} \phi_t(u) \in \mathcal{M}^*$ Impossible ! $\Rightarrow \forall\ t \in \mathbb{R}_+ \quad \forall\ A \in \mathcal{A}\ \phi_t(A) \in \mathcal{A}$

Take $A \in \mathcal{A}$ s.t. $\max_{u \in A} E(u) < \beta + \varepsilon_0/2$. Apply ϕ_t...cont. ! $\square$

Birkhoff Existence Result Revisited.

Birkhoff Existence Result Revisited.

$\mathcal{M} := W^{1,2}(S^1, N^2)$ where $N^2 \simeq S^2$

Birkhoff Existence Result Revisited.

$\mathcal{M} := W^{1,2}(S^1, N^2)$ where $N^2 \simeq S^2$defines a *complete Finsler manifold*.

Birkhoff Existence Result Revisited.

$\mathcal{M} := W^{1,2}(S^1, N^2)$ where $N^2 \simeq S^2$defines a *complete Finsler manifold*.

E is (PS) on $\mathcal{M}$.

Birkhoff Existence Result Revisited.

$\mathcal{M} := W^{1,2}(S^1, N^2)$ where $N^2 \simeq S^2$defines a *complete Finsler manifold*.

E is (PS) on $\mathcal{M}$.

Let any sweep-out $\vec{\sigma}_0$ of N^2 corresponding to a non zero element of $\pi_2(N^2)$.

Birkhoff Existence Result Revisited.

$\mathcal{M} := W^{1,2}(S^1, N^2)$ where $N^2 \simeq S^2$defines a *complete Finsler manifold*.

E is (PS) on $\mathcal{M}$.

Let any sweep-out $\vec{\sigma}_0$ of N^2 corresponding to a non zero element of $\pi_2(N^2)$.$\mathcal{A} := \Omega_{\vec{\sigma}_0}$ is admissible.

Birkhoff Existence Result Revisited.

$\mathcal{M} := W^{1,2}(S^1, N^2)$ where $N^2 \simeq S^2$defines a *complete Finsler manifold*.

E is (PS) on $\mathcal{M}$.

Let any sweep-out $\vec{\sigma}_0$ of N^2 corresponding to a non zero element of $\pi_2(N^2)$.$\mathcal{A} := \Omega_{\vec{\sigma}_0}$ is admissible.

Palais Theorem

Birkhoff Existence Result Revisited.

$\mathcal{M} := W^{1,2}(S^1, N^2)$ where $N^2 \simeq S^2$defines a *complete Finsler manifold*.

E is (PS) on $\mathcal{M}$.

Let any sweep-out $\vec{\sigma}_0$ of N^2 corresponding to a non zero element of $\pi_2(N^2)$.$\mathcal{A} := \Omega_{\vec{\sigma}_0}$ is admissible.

Palais Theorem $\Rightarrow$

$$W_{\vec{\sigma}_0} = \inf_{\vec{\sigma}\in\Omega_{\vec{\sigma}_0}\cap\Lambda} \max_{t\in[0,1]} E(\vec{\sigma}(t,\cdot)) > 0$$

is achieved by a **closed geodesic**.

Birkhoff Existence Result Revisited.

$\mathcal{M} := W^{1,2}(S^1, N^2)$ where $N^2 \simeq S^2$defines a *complete Finsler manifold*.

E is (PS) on $\mathcal{M}$.

Let any sweep-out $\vec{\sigma}_0$ of N^2 corresponding to a non zero element of $\pi_2(N^2)$.$\mathcal{A} := \Omega_{\vec{\sigma}_0}$ is admissible.

Palais Theorem $\Rightarrow$

$$W_{\vec{\sigma}_0} = \inf_{\vec{\sigma}\in\Omega_{\vec{\sigma}_0}\cap\Lambda} \max_{t\in[0,1]} E(\vec{\sigma}(t,\cdot)) > 0$$

is achieved by a **closed geodesic**.

This gives a new proof of **Birkhoff existence result**.